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# THE ENUMERATION OF LOCALLY TRANSITIVE TOURNAMENTS

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The enumeration of locally transitive tournaments

by

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ABSTRACT

A tournament is locally transitive if for each vertex x both $\Gamma^+(x)$ and $\Gamma^-(x)$ are transitive tournaments. We establish an isomorphism between such objects and shift registers where the complement of the bit shifted out of the last position is shifted into the first position. As a consequence the number of locally transitive tournaments on n vertices is found to be

$$\sum_{d|n} \frac{2^{d-1}}{d} \text{ odd } \left(\frac{n}{d}\right) \sum_{e|\frac{n}{d}} \frac{\mu(e)}{e}$$

where $\mu$ is the Möbius function and odd (i) is one or zero according to whether i is odd or even.

KEY WORDS & PHRASES: *enumeration, tournament, shift register, colour scheme*

## 0. INTRODUCTION AND MOTIVATION

In a lecture given at the 16th Dutch Mathematical Congress Peter Cameron discussed colour shemes. Let me repeat some fragments of his talk.

Let X be a set of cardinality $n \geq 2k+1$. Colour $\binom{X}{k}$ (the collection of all k-subsets of X) with r colours $c_1,\ldots,c_r$. The *colour scheme* of a (k+1)-set is the vector $(a_i)_{1\leq i\leq r}$ where $a_i$ is the number of k-subsets of this (k+1)-set with colour $c_i$. (Thus $\Sigma\, a_i = k+1$.) The *colour scheme matrix* A of such a colouring has as its rows the distinct colour schemes.

<u>THEOREM</u>. *# of colour schemes ≥ # of colours.*

Cameron proceeded to discuss the case of equality and derived several results for large $|X|$.

<u>THEOREM</u>. *If* $|X|$ *is large with respect to* k *and* r *then* A *is triangular.*

<u>CONJECTURE</u>. *If* $|X|$ *is large w.r.t.* k *then* A *is triangular.*

A colour scheme matrix is called *stable* if there exist corresponding colourings for arbitrarily large n. As an example he analysed the stable colour scheme matrices in the case $k = 3$, $r = 2$.

In fact, dropping the restriction that A be stable we have the following possibilities:

(i) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

No example exists. [If $n \leq 4$ then at most one colour scheme occurs. If $n \geq 5$ then by removing points if necessary we may suppose $n = 5$. Taking complements we see an A-colouring implies a decomposition of $K_5$ into two subgraphs with all valencies odd, which is clearly absurd.]

(ii) $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

No example exists: if all colour schemes are either (4 0) or (0 4) then only one colour scheme occurs.

(iii) $A = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$.

(Let the colours be red and blue.) With induction on n one proves easily that if not all triples are red then there is a point $x_0$ such that the blue triples are exactly those containing $x_0$.

(iv) $A = \binom{4\ 0}{3\ 1}$.

Here the blue triples form a collection of triples no two of which intersect in more than one point, i.e., a partial Steiner triple system.

(v) $A = \binom{3\ 1}{2\ 2}$.

The only example with $n \geq 7$ has $n = 7$ and is isomorphic to the following:

$X = \mathbb{Z}_6 \cup \{\infty\}$. Red triples: $\{\infty,0,3\}$, $\{\infty,0,2\}$, $\{0,1,2\}$, $\{0,1,3\}$ (mod 6), 21 in all.

Blue triples: $\{\infty,0,1\}$, $\{0,1,4\}$, $\{0,2,4\}$ (mod 6), 14 in all.

In other words: the blue triples form the unique 2-(7,3,2) design. [This is seen as follows: fix a point, say $\infty$. Colour the pairs $pq$ in $X\setminus\{\infty\}$ with the colour of $\infty pq$. Look at the graph of the blue edges. The conditions are: on a blue triple the graph has at most one edge, on a red triple the graph has one or two edges. It follows that the graph does not contain triangles, 3-claws, 4-circuits etc. There are examples with $5 \leq n \leq 7$.]

[Remark: without this detailed analysis one can at least say immediately that $n$ is bounded by the Ramsey number $N(4,4;3)$, i.e., this case is not stable.]

(vi) $A = \binom{4\ 0}{2\ 2}$.

This case is slightly more complicated than the previous ones. It is the special case of a two-graph where no (0 4) occurs. Let $x \sim y$ if $x = y$ or $\{x,y\}$ not contained in any blue triple. Then $\sim$ is an equivalence relation and moreover $x \sim y$ implies that $xuv$ and $yuv$ have the same colour for all $u,v \neq x,y$. Therefore we may restrict ourselves w.l.o.g. to the case of *reduced* colourings, those where all equivalence classes have size one. Now we have:

PROPOSITION *A reduced A-colouring is one of*

(i) *a locally transitive tournament on X, where the blue triples are those carrying a 3-cycle*

(ii) *a unique example with* 6 *points: the* 2-(6,3,2) *design; only* (2 2) *occurs.*

[PROOF. Fix a point $\infty$ and construct a blue graph as before. It does not contain $K_3$ or $2K_2$. If it is not bipartite, it contains a pentagon. If it is a pentagon we are in case (ii). Otherwise there must be more points and edges and we find $2K_2$ unless the graph is bipartite. Now apply induction on n: on $X\setminus\{\infty\}$ we have a locally transitive tournament, and $X\setminus\{\infty\} = V_1 + V_2$ where $V_1$ and $V_2$ are independent in the blue graph, i.e. all triangles inside $V_1$ or $V_2$ are red. Now it is easy to see that there is a unique way to extend the tournament to X, by having all arrows $(u\ \infty)$ for $u \in V_1$ and $(\infty\ u)$ for $u \in V_2$ (or vice versa).]

[REMARK: Case (ii) is not really an A-colouring, but this case must be included since reducing an A-colouring could remove all (4 0) colour schemes.]

Here Cameron asked for the number of nonisomorphic locally transitive tournaments and remarked that since $\lceil 2^{n-1}/n \rceil$ is the right answer for $1 \le n \le 8$ and for n a power of two, it might be the answer for all n. This motivated the present investigation.

First of all I noticed that the only sequence in Sloane's handbook of integer sequences starting with 1,2,2,4,6,10,16 continued with 30,52,94, where the above formula would give 29,52,94. This engendered some doubt as to the correctness of the formula, and in fact the true answer coincides with Sloane's sequence #121. [This sequence is labeled "shift registers" and Sloane provides a reference to a book by Golomb which is not easily available to me; but since we prove isomorphism between locally transitive tournaments and certain shift registers ( and since the first eleven terms agree) I am convinced that our sequence is the one intended by Sloane. In table 1 the first thirty terms are listed.]

## 1. LOCALLY TRANSITIVE TOURNAMENTS

A *tournament* is a directed graph on n vertices without loops or circuits of length two such that the underlying undirected graph is $K_n$ - in other words, if $x \neq y$ then it has exactly one of the edges xy and yx. A *transitive* tournament is a tournament such that $xy \in \Gamma$ and $yz \in \Gamma$ implies $xz \in \Gamma$, where $\Gamma$ is its collection of edges. A locally transitive tournament is a tournament such that the subtournaments $\Gamma^+(x) := \{y | xy \in \Gamma\}$ and $\Gamma^-(x) = \{y | yx \in \Gamma\}$ are transitive. (In other words, $\Gamma^-(x) \cup \{x\}$ and $\{x\} \cup \Gamma^+(x)$ are linearly ordered sets.) Now let $(X,\Gamma)$ be a locally transitive tournament.

A. Let $a \in \Gamma^+(x)$. Then $\Gamma^+(a)$ is the union of a terminal interval in $\Gamma^+(x)$ and an initial interval in $\Gamma^-(x)$. [For: suppose $b,c \in \Gamma^-(x)$, $bc \in \Gamma$, $c \in \Gamma^+(a)$, $b \notin \Gamma^+(a)$. Then $c,x,a \in \Gamma^+(b)$ and we have the edges $cx,xa,ac$, a contradiction.]

B. X can be ordered cyclically such that for each $a \in X$ the sets $\Gamma^-(a) \cup \{a\}$ and $\{a\} \cup \Gamma^+(a)$ are intervals in the cyclic order (with end point a).

C. Introduce n new objects $a'$ for $a \in X$ such that $X \cup X'$ is ordered cyclically, the restriction of the cyclic order to X is the one we had under B and such that $\Gamma^-(a) = (a',a) \cap X$ and $\Gamma^+(a) = (a,a') \cap X$. (That is, the objects $a'$ indicate the boundary between $\Gamma^-(a)$ and $\Gamma^+(a)$.) If b and c are adjacent points in X then $\{x' | b < x' < c\}$ is ordered by $x' < y'$ iff $x < y$.

Now we have: If $a \neq b$ then the pair $a,a'$ separates the pair $b,b'$ in the cyclic order.

[For: suppose not. Then w.l.o.g. $a < b < b' < c < a' < a$ in the cyclic order. But $\Gamma^+(a)$ contains b and c, and the cyclic order restricted to $\Gamma^+(a)$ is the linear order on $\Gamma^+(a)$ so there is an edge bc contradicting $b < b' < c$.]

D. Starting at an arbitrary point in the cycle $X \cup X'$ label the points $1,2,\ldots,n,1,2,\ldots,n$ where a label gets a prime if the corresponding point is in $X'$.

[This is appropriate, since between a and $a'$ there is exactly one of b and $b'$ for any point $b \neq a$, so a and $a'$ have distance n in the cyclic order.]

E. Now the labels themselves do not carry information, that is, we can encode the sequence 12'34'5'... in binary 01011... , writing 0 for points in X and 1 for points in $X'$.

Since all steps are 1-1 we have proved the following:

*There is a* 1-1 *correspondence between isomorphism classes of locally transitive tournaments and classes of binary vectors* $\underline{v}$ *of length* 2n *such* *that* $v_{i+n} = 1-v_i$ $(1 \le i \le n)$, *where the classes are the collections of all cyclic shifts of their members.*

[In terms of shift registers, a class can be seen as the collection of values of a shift register of length n wired in such a way that the bit shifted into the first position is the complement of the bit shifted out of the last position.]

Now examine the sizes of the various classes, i.e., the length of the orbits of our vectors of length 2n under the cyclic groups of order 2n. Most of the classes will have full size 2n, and the remaining ones are smaller, so that

$$\frac{2^n}{2n}$$

is a lower bound for the total number of classes. Also, if $d|n$ then an orbit of size d does not occur since our vectors are not invariant under shifting over n positions. (But if n is a power of two, then any proper divisor of 2n is a divisor of n, i.e., $2^n/2n$ is the correct answer in this case.) Consequently all orbits have even length.

For 2d/2n, 2d+n (i.e., $\frac{n}{d}$ odd) let $N_d$ be the number of orbits of size 2d. Note that $N_d$ does not depend on n: it is the number of vectors of length 2d with $v_{i+d} = 1-v_i$ such that their orbit under the cyclic group of order 2d has size 2d.

Obviously

$$\sum_{d|n} 2d\ N_d.\mathrm{odd}(\tfrac{n}{d}) = 2^n.$$

Using Möbius inversion one finds

$$2n\ N_n = \sum_{d|n} 2d\ N_d\ \mathrm{odd}(\tfrac{n}{d}) \sum_{d|m|n} \mu(\tfrac{n}{m})$$

$$= \sum_{m|n} \mu(\tfrac{n}{m})\ \mathrm{odd}(\tfrac{n}{m}) \sum_{d|m} 2d\ \mathrm{odd}(\tfrac{m}{d})\ N_d$$

$$= \sum_{m|n} \mu(\tfrac{n}{m})\ \mathrm{odd}(\tfrac{n}{m})\ 2^m.$$

The total number of orbits is

$$N = \sum_{d|n} \mathrm{odd}(\tfrac{n}{d}) \,.\, N_d$$

$$= \sum_{m|n} \frac{2^m}{2m} \mathrm{odd}(\tfrac{n}{m}) \sum_{e|\frac{n}{m}} \frac{\mu(e)}{e} \,.$$

Table 1 - the number of nonisomorphic locally transitive tournaments

| n | N(n) | N(n+10) | N(n+20) |
|---|---|---|---|
| 1 | 1 | 94 | 49940 |
| 2 | 1 | 172 | 95326 |
| 3 | 2 | 316 | 182362 |
| 4 | 2 | 586 | 349536 |
| 5 | 4 | 1096 | 671092 |
| 6 | 6 | 2048 | 1290556 |
| 7 | 10 | 3856 | 2485534 |
| 8 | 16 | 7286 | 4793492 |
| 9 | 30 | 13798 | 9256396 |
| 10 | 52 | 26216 | 17895736 |

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{Sequence #121: "shift registers" - 1,2,2,4,6,10,16,30,52,94 - ref[2].}

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